$$\frac{p_{m_1n}(A'_{m+n}-A_{m+n})}{r^{v}}.$$

Perhaps the most equitable conclusion would be this, that when conditions have been violated *without fraud,* the policy should be paid under the above deduction.

On the Discovery of the Law of Human Mortality, and on the antecedent partial Discoveries of Dr. Price and Mr. Gompertz. By T. R. EDMONDS, B.A., *formerly of Trinity College, Cambridge.*

IN the year 1832 there was published, in my name, an extensive collection of *Life Tables,* founded upon the discovery of the law which, in my belief, governs the mortality, according to age, of all nations and classes of men, from the earliest infancy to extreme old age. In these tables the numbers living or surviving at successive ages have been deduced from a simple formula expressing the proportion of survivors at any age in terms of the mortality. They are the first Tables of this kind ever published, and they have been used daily in the practice of life assurance, and in the valuation of life contingencies, for a period of thirty-two years, including four years before publication.

The following is the law of human mortality:—The whole duration of human life is divided into three well-marked stages, which belong to all animal life, and are—stages of growth, maturity, and decay. The mortality, in all three stages, increases or decreases, uniformly with the age, in geometrical progression, but in a different progression for each of the three stages. The constant ratios of progression belonging severally to the three stages have been ascertained, their values being the same for all populations at the same stages of life.

The three stages of human life may be conveniently designated as those of *infancy, florescence,* and *senescence.* The period of infancy most commonly ends at the age of 9 years; the period of senescence most commonly begins at the age of 55 years. The values of the annual constants of progression in the stages of infancy, florescence, and senescence respectively, are $\frac{1}{1{\cdot}479108}$, 1·0299117, and 1·0796923—the logarithms of these numbers being $-{\cdot}1700$, $+{\cdot}0128$, and $+{\cdot}0333$ respectively. If the force

of mortality continued uniform for one year, at the beginning of any of the three stages of life, be represented by (a), and if the constant annual ratio of geometrical progression be represented by (p), we shall have, for any time or age (x) measured from the beginning of the stage, (ap^x) to represent the force of mortality at that time; the same being true for all values of (x), whether (x) is a whole number or a fraction. When the mortality has been ascertained by observation for any age, and when the exact length of the period of florescence is known, the mortality existing at every other age may be calculated from the formula (ap^x); the quantity (p) having one or other of the three values given above, according to the stage of life observed.

The limits of the periods of infancy and florescence vary slightly from the ages of 9 and 55 years respectively, as well in different populations as in the same population at different epochs. In every population the force of mortality, or the ratio of the dying to the living, is always least at the age or short period immediately preceding the commencement of puberty—say from the age of 9 to the age of 11 years, or from the age of 8 to the age of 12 years. In my principal theoretical tables, I have assumed the existence of a short period, from the age 8 to the age 12 years, wherein the mortality is constant and at a minimum. This assumption is in accordance with appearances presented by large numbers attaining severally their lowest mortality at different ages—near 9 years of age—although each individual may suddenly pass out of the first into the second stage without passing through the supposed intermediate stage.

The first intimation of the nature of the law of human mortality was conveyed to the public in an essay of Dr. Price, read at the Royal Society in April, 1769. This essay is printed in *Price's Observations on Reversionary Payments*, a work which was for many years the chief text-book used by students of the valuation of life contingencies. In this essay Dr. Price made the following remark (page 400, vol. ii., edition 7) on the mortuary register of the parish of Holy Cross, near Shrewsbury:—"This register exhibits, with remarkable regularity and consistency, the progress of human mortality from birth to old age—representing human life, in conformity to other observations, as particularly weak in the first month, and from that age as growing gradually stronger, till at 10 it acquires its greatest strength, which it afterwards loses; but more slowly till 50, and after 50 more rapidly, till at 70 or 75 it is brought back to all the weakness of the first month."

In the above statement may be recognized the chief features of the true law of human mortality. The three stages of human life are designated, their limits are noted, and words are used descriptive of the effect of the different constants belonging to the three separate stages. Dr. Price, to perfect his discovery, would have had to make the following additional remark, viz.:—"That the rate of increase or decrease of the mortality was constant throughout each of the periods designated; that life growing *gradually stronger* was represented by the constant annual ratio of 1·479 to 1; that life losing strength *more slowly* was represented by the constant annual ratio of 1 to 1·0299; and that life losing strength *more rapidly* was represented by the constant annual ratio of 1 to 1·0797.

In the discovery of the law of human mortality, I was not assisted by any knowledge or recollection of the antecedent partial discoveries of Dr. Price or any other writer. My discovery is founded on direct observation of the principal and best-established facts or statements of mortality on record. These facts, collected by me and expressed in their simplest form, denoted observed ratios of dying to living at successive equal intervals of age. At ages above 10 years, the mortality at successive quinquennial intervals of age was thus ascertained from an extensive variety of tables and observations. These successive quinquennial rates seldom indicated any regularity of increase; but, when combined so as to form rates of mortality for successive decennial intervals of age, the uniform rates of progressive increase became manifest, and the two numbers already indicated, as regulating the increase in the periods of florescence and senescence, were discovered. The constant ratio regulating the progressive decrease of the mortality during the period of infancy, was found by observations on the mortality exhibited at this period in a large variety of tables for successive annual and biennial intervals of age. The constant of the period of infancy was found to be the same in all tables and observations of mortality; so also the constant of the period of senescence was found to be the same in all observations. The constant of the period of florescence was found, however, occasionally to vary from its average magnitude, when small sections of a population were observed; but in large aggregates, consisting of mixed population, and in small sections, consisting of homogeneous population, the particular constant already indicated was manifested.

Immediately after the discovery of the law of human mortality, I constructed, for practical use, a table of mortality in which the numbers living or surviving at successive annual intervals of age

were deduced by successive subtractions of the approximate annual decrements obtained from the formula $\Delta y = y \times ap^x \Delta x$. By this table was exhibited a series of annual decrements uniformly progressive with age, excepting at very advanced ages, when the uniformity was interrupted, although the defect was of no importance in practice. By reflection on this defect, manifested at advanced ages, I became convinced that uniformity of decrement according to age could not be obtained by means of an annual decrement of the form $\Delta y = y \times ap^x \Delta x$; and that such uniformity could only be secured by continually diminishing the intervals of age observed, from Δx to dx, or by calculating the decrement in infinitely small intervals of age by the formula $dy = -y \times ap^x dx$.

Until I had arrived at the conclusion that the decrement of the living at any age was of the form $dy = -y \times ap^x dx$, the idea had never occurred to me that there existed anything in common between the formula (g^{p^x}), used by Mr. Gompertz to represent the number living, and my formula (ap^x)—with three permanent values of (p) determined, representing the decrement of life for every year of age. I had previously become acquainted with the formula of Mr. Gompertz by conversation with Mr. Gompertz himself; I had also tried the value of his formula by applying it to interpolation of the numbers living, in a table of mortality of my own selection, and found the result not to be in conformity with Mr. Gompertz's verbal statement. Not believing his theory to be well founded, I soon forgot everything relating to it, except that his formula was (g^{p^x}), and that he used the differential calculus in his investigation, either in descending from the integral to the differential, or in ascending from the differential to the integral. After the true law of human mortality had become known to me, on re-examination of Mr. Gompertz's theory, I became aware of the cause of my original failure to find any truth in his representations. The error was on the side of Mr. Gompertz, who, in his statement to me, made no mention of any limits of age circumscribing the application of his formula. In all probability, at my first examination of his formula, I chose for interpolation two numbers representing the survivors at two ages, on different sides of one of the limits. The result of the comparison would, in this case, evidently be contradictory to Mr. Gompertz's theory.

The statement of his theory made to me does not differ materially from the general statement given by Mr. Gompertz in his paper read at the Royal Society in June, 1825, for he there states it to be a mathematical consequence of his formula being found to

be true by observation, "That the average exhaustions of a man's power to avoid death are such that, at the end of equal infinitely small intervals of time, he loses equal portions (or proportions) of his power to oppose destruction." As no limits of age are here mentioned, it might fairly be inferred from the above statement, that the vital force of man, measured by the ratio of the living to the dying, is in a constant state of decay from birth to the end of life, at one and the same uniform rate. Mr. Gompertz nowhere makes mention of the fact that the vital force increases at a high rate during the period of infancy; nor does he note the existence of any difference between the rate of the annual loss of vital force during the period of florescence, and the greater rate of annual loss of vital force during the period of senescence. All the examples in detail which he adduces in support of his theory are confined to specimens, taken from different tables, of numbers surviving at various ages, between 15 and 55 years, or comprehended in one stage of life only, that of florescence. With respect to ages above 55 years, Mr. Gompertz states the result of his examination of one table of mortality only, without entering into detail as in the other cases.

My knowledge of the law of human mortality is founded on direct observation of the ratios of the dying to the living, at successive intervals of age, in various populations. Mr. Gompertz's knowledge of the law has been obtained indirectly, being founded on compilations made by others, of ratios of dying to living. The laws of mortality in the last century of the populations of Sweden and of Carlisle are contained in the observed ratios of dying to living at various ages, collected and published by Wargentin and Heysham. Mr. Gompertz, in deducing the laws of mortality of these populations, relies upon tables of survivors, according to age, constructed by Price and Milne, who acted as compilers of the ratios of mortality supplied by Wargentin and Heysham. He accepts observations at second hand for his guide when the original ratios of mortality observed were easily accessible. Mr. Gompertz has done nothing without the aid of tables of survivors at successive ages previously constructed by others. Nearly all that he offers to show is, how *interpolations* may be made for intermediate ages when the number of survivors at the beginning, and the number of the survivors at the end, of a large interval of age are given. Mr. Gompertz never shows how a new table of mortality may be constructed independently of other previously-existing tables.

The paper of Mr. Gompertz read at the Royal Society in June, 1825, is entitled, "On the nature of the function expressive of the law of human mortality, and on a new mode of determining the value of life contingencies." This paper is divided into two chapters, the first containing 17 pages, and the second containing 58 pages. The subject of both chapters is *interpolation* for intermediate ages between the number of survivors at the beginning, and the number of survivors at the end, of intervals of age, of greater or less extent, given by different tables of mortality. Chapter I., or rather 12 out of its 17 pages, is occupied in showing that *interpolation* may be most correctly effected by the aid of the transcendental (g^{p^x}). The whole of Chapter II. is devoted to showing that interpolations for short intervals of age may be satisfactorily effected by means of a simple geometrical series, of which each term is of the form (g^x). As the two modes of interpolation are, in a certain degree, inconsistent with and opposed to one another, it is not easy to account for the publication of these two chapters side by side. The reader might fairly infer that the author could not have much faith in the virtues of his transcendental (g^{p^x}), when, in the inculcation of its truth and applicability, he had bestowed only one-fifth of the space and attention which he had bestowed on the simple and common geometrical series represented by (g^x).

If the two chapters are viewed together, the reader can hardly avoid coming to the conclusion that the patient labour expended in the production of Chapter II. was the source of the discovery exhibited in Chapter I., and that the ideas connected with the discovery have been presented to the public in an order which is the reverse of the order in which they occurred to the mind of Mr. Gompertz. If anyone acquainted with mathematical formulæ, and with the transcendental (g^{p^x}), had expended, with tolerable success, a vast amount of labour (as Mr. Gompertz apparently had done) in the interpolation of survivors according to age, by means of the geometrical form (g^x), it could hardly have failed to occur to his mind, in the course of his labour, that (g^{p^x}) being a form of much greater elasticity than (g^x), would yield numbers much more nearly coinciding with the facts intended to be represented. After he had, by examination, found his prognostication verified, he would not (at first, at least) regard the quantity (p) as a permanent quantity; for, since (g^{p^x}) is taken only as the substitute of (g^x), and as (g) in (g^x) has an endless number of values, the quantity (p) would also be considered as having an endless number of values. It

would be supposed that (p) would change with every different interpolation, because (g) in (g^x) changes with every different interpolation.

As soon as Mr. Gompertz had discovered that the transcendental (g^{p^x}), when used for interpolation of survivors according to age, gave results nearly coincident with fact, his curiosity would be awakened as to the signification of the constituent parts of (g^{p^x}). The readiest means of examination would be afforded by differentiating the above quantity. The differential obtained would be found to be $\log. g \times \log. p \times g^{p^x} p^x dx$, which is of the same form as $dy = a \times yp^x dx$, the differential arrived at by my investigation previously to any knowledge of its connexion with (g^{p^x}) as its integral. Thus Mr. Gompertz will have descended from his integral to my differential—there being no reason to suppose that he had ascended, from a differential like mine, to his integral. The former course is simple and obvious; but the latter course, though apparently simple, would not be obvious to a person dealing for the first time with quantities of a new and unknown character. It is often erroneously supposed that Mr. Gompertz has deduced his formula from the hypothesis of equal proportions of vital force being lost in equal intervals of age; but the reverse is the fact, the hypothesis having been arrived at as the mathematical or necessary consequence of the truth of the formula; for Mr. Gompertz states that "the hypothesis was derived from an analysis of the experience alluded to"—the experience alluded to consisting in (g^{p^x}) being found to yield correct interpolated numbers for the survivors according to age in many tables of mortality.

The formula (g^{p^x}) is a general formula, alike applicable to the expression of the true law of human mortality and to the imperfect law announced by Mr. Gompertz. The true law differs from the imperfect law only in respect of the value of (p) being determined in the former case, and being left undetermined in the latter case. According to the true law of human mortality, (p) has a fixed determined value for each of the three periods or stages into which human life is divided—growth, maturity, and decay—for the same and for different populations. According to the imperfect law of mortality declared by Mr. Gompertz, no limit is assigned to the variation of (p), whether in the same or different populations, at any period of life. The original idea entertained by Mr. Gompertz apparently was, that by means of the formula (g^{p^x}), interpolations of survivors between any pair of ages, not very remote from one another, may be effected by assuming different values of (p). It

is only by reference to examples of interpolation cited by Mr. Gompertz that the reader learns, indirectly, that, in the second period of life, or from the age 15 to 55 years, Mr. Gompertz regards (p) as constant in the same population at the same time, though variable for different populations and for the same population at different times. Mr. Gompertz says nothing of the value of (p) at any age under 15 years, and gives the value of (p) in one case only for the period of life above 55 years of age. In examples of the application of his formula to different populations, Mr. Gompertz always treats (p) as a fugitive quantity, and as of no value except for the particular interpolation then required to be made.

The view now taken of Mr. Gompertz's theory of mortality, with reference to the variability of the quantity (p), has evidently been also taken by the late Mr. Galloway, who, I believe, is the only person who has made a practical use of Mr. Gompertz's formula in the construction or adjustment of a new table of mortality. In the year 1841, whilst explaining the construction of tables which he had made of the mortality experienced by the Amicable Assurance Society, Mr. Galloway states that, in the adjustment of these tables, he "had recourse to the highly ingenious method (founded, indeed, on a hypothetical principle) proposed by Mr. Gompertz. . . . . In Table III. the same constants were used through the whole series, from age 45 to age 93 years. In Table IV., from age 24 to age 68, one set of constants was used; and a second set from age 69 to age 93 years." Mr. Galloway does not state the value of the quantity (p) in either of the three cases, nor does he make any allusion to (p) as distinguished from other but less important constants used. He evidently regarded (p) as a fugitive quantity, of no value except for the use to which it had been applied, in smoothing the irregularities which would otherwise have appeared in the tables which he was then constructing. Both Mr. Gompertz and Mr. Galloway have treated (p) as an insignificant quantity, of accidental value, and of no use but to maintain the formula (g^{p^x}). The truth is, however, that (p), with its three determinate values, is independent of all formulæ, has existed as long as man has existed, and forms part of the foundations of the universe.

All my published tables of mortality have been deduced from the formula $y=10^{\frac{k^2a}{\lambda p}(1-p^x)}$. In arriving at this formula, I am not conscious of having received any assistance from the writings of

Mr. Gompertz, beyond the suggestion of the idea that the form which I had discovered for the decrement of human life at every age was one capable of integration by simple and well-known methods. I do not believe that this assistance was of any importance, except with regard to time, for I possessed all the materials for forming the series proposed for summation, and could hardly have failed to discover the law of that series as soon as my attention had been directed to, and concentrated on, the subject. The law of the series may be found as well without as with the aid of the differential calculus. My formula, as given above, is reduced to more simple terms than that given by Mr. Gompertz. The correctness of my formula, or of the process by which it has been obtained, has never been questioned. There is, however, ground for doubting the correctness of the process by which Mr. Gompertz has deduced his formula. To remove this doubt, the publication of his process in a more explicit form is requisite.

My formula, $y=10^{\frac{k^2a}{\lambda p}(1-p^x)}$, gives the number living or surviving at any age (x) in any of the three periods of human life, expressed in terms of the mortality (ap^x) at that age, and known quantities; (k) being the modulus of the common system of logarithms and equal to ·4342945, and (λp) being the common logarithm of (p), the constant ratio of the period. This formula may be written thus:—

$$y=10^{c(1-p^x)}=g^{1-p^x}=\frac{g}{g^{p^x}}.$$

Mr. Gompertz, by changing the sign of (c) and introducing the superfluous quantity (d), makes his corresponding formula $y=dg^{p^x}$. If the change of sign had not been made, Mr. Gompertz would have stated the value of (y) to be $\frac{d}{g^{p^x}}$, instead of $\frac{g}{g^{p^x}}$, my value given above. According to my formula, $y=1$ when $x=0$, whilst Mr. Gompertz's formula gives $y=\frac{d}{g}$ when $x=0$. The integration of Mr. Gompertz's formula requires to be performed between limits of $x=a$ and $x=a+n$, whilst in my formula the limits are of the simplest form between 0 and x. The defect in Mr. Gompertz's formula, caused by the addition of (d), is the same as that which would exist in a table of discount of money at compound interest if any other basis were adopted than the value of the sum of £1 receivable (x) years hence. It is only in cases of interpolation that the quantity (d) is of any utility.

It may be useful here to give an example of the application of my formula $y_x = \frac{g}{g^{p^x}}$; for this purpose we will assume (x) to be equal to 10 years, and to be measured from the beginning of the second and chief of the periods of human life—the period of florescence. In this period $p = 1{\cdot}0299117$, and the value of (a), or the minimum mortality, is, in most populations, on an average, $= {\cdot}0063643$; hence $\frac{k^2a}{\lambda p}$ or $c = {\cdot}0937799$, and 10^c or $g = 1{\cdot}241023$. Since $p^{10} = 1{\cdot}342765$, we get $g^{p^{10}} = g^{1{\cdot}342765} = 1{\cdot}336363$;

whence $$y_{10} = \frac{g}{g^{p^{10}}} = \frac{1{\cdot}241023}{1{\cdot}336363} = \frac{1}{1{\cdot}076823};$$

otherwise,

$$y_{10} = g^{1-p^{10}} = g^{1-1{\cdot}342765} = g^{-{\cdot}342765} = \frac{1}{g^{{\cdot}342765}} = \frac{1}{1{\cdot}076823},$$

as before.

The formula above given, when expressed in logarithms, becomes $\lambda y_x = \frac{k^2a}{\lambda p}(1 - p^x)$, whence may be derived the logarithms of the probability of living one year from the age (x), — quantities by the aid of which the construction of mortality tables is most facilitated. The logarithm of such probability is equal to the logarithm of the value of (y) at age $(x+1)$, less the logarithm of the value of (y) for the age (x).

Since—

$$\lambda y_x = \frac{k^2a}{\lambda p}(1 - p^x), \text{ and } \lambda y_{x+1} = \frac{k^2a}{\lambda p}(1 - p^{x+1}),$$

we get, by subtraction,

$$\lambda y_{x+1} - \lambda y_x = -\frac{k^2a}{\lambda p}(p^{x+1} - p^x) = -\frac{k^2a}{\lambda p} \times (p-1)p^x,$$

$$\text{or } \lambda\frac{y_{x+1}}{y_x} = -mp^x; \quad \text{putting } m = \frac{k^2a}{\lambda p}(p-1).$$

That is to say, the logarithm (which is negative) of the probability of living one year from the age (x) is equal to (p^x) multiplied by the constant quantity $\frac{k^2a}{\lambda p}(p-1)$.

If, instead of the common or decimal system of logarithms, having 10 for its base, we use the hyperbolic system, of which the

base is $e=2{\cdot}7182818$, we shall get, for the hyperbolic logarithm of the probability of living one year,

$$l\frac{y_{x+1}}{y_x}=-\frac{a}{\beta}(p-1)p^x,$$

$$=-\frac{a}{\beta}\left(\beta+\frac{\beta^2}{1.2}+\frac{\beta^3}{1.2.3}+\&c.\right)p^x,$$

$$=-a\left(1+\frac{\beta}{2}+\frac{\beta^2}{1.2.3}+\&c.\right)p^x;$$

(β) representing the hyp. log. of (p), and (p) being developed in terms of its logarithm, (p) or e^β being $=1+\beta+\frac{\beta^2}{1.2}+\frac{\beta^3}{1.2.3}+\&c.$: also ($\beta$) bearing the same relation to ($p-1$) that (a) bears to (a). The last quantity (a) measures the decrement in one year of age suffered in passing through that year by a given number ($1+a$) of persons alive at the beginning of the year, when the mortality is constant or when (p)$=1$.

The equation $y=\frac{g}{g^{p^x}}=g^{-p^x+1}$ having been obtained by integration, it may be useful to reverse the process and show the steps by which descent is made from the integral to the differential.

We have—

$$dy=g\times d.g^{-p^x}=-g\times g^{-p^x}\times lg\times lp\times p^x dx,$$

$$=-g^{-p^x+1}\times\frac{\lambda g}{k}\times\frac{\lambda p}{k}\times p^x dx,$$

$$=-y\times\frac{k^2 a}{\lambda p}\times\frac{\lambda p}{k^2}\times p^x dx,$$

$$=-y\times a\times p^x dx,$$

which is the original differential equation.

In deducing his formula, $y=d.g^{p^x}$, Mr. Gompertz begins from $dy=-y\times ap^x dx$, then states, as a consequence, that $\frac{dy}{y}=-abp^x dx$ [a new quantity, (b) being introduced as arising in the process of integration], and afterwards concludes that $y=d.g^{p^x}$, (g) being stated to be the number whose common logarithm is (c), and (c) being stated to be equal to the common logarithm of $\frac{1}{p}$ multiplied by the square of the hyperbolic logarithm of 10. Since (d), a particular value of (y), does not express or exhibit (a) or (b),

the result arrived at is the extraordinary one, that the formula gives the number living or surviving at the age (x) in terms entirely independent of the mortality at that or any other age. In the last Number of the *Assurance Magazine* (July, 1860), Mr. De Morgan, in his office of self-constituted judge between Mr. Gompertz and me, overlooks this important error, although he corrects other errors of inferior importance. A very simple process of integration has been rendered obscure and ambiguous by the aid of two superfluous and useless indeterminate constants, (b) and (d). The imperfect corrections of Mr. De Morgan have not diminished the original obscurity and ambiguity.

Mr. De Morgan makes the following remark respecting the introduction by Mr. Gompertz of the quantity (b):—"The preceding (b) is a superfluous quantity, which is useless. It seems to have originated in the idea that the diminution of the number living is proportional to the intensity of mortality, and therefore represented by that intensity multiplied by a constant. But the constant (a) already introduced is sufficient." On this opinion of Mr. De Morgan I would observe that a factor (b), which is greater or less than unity, cannot be superfluous and useless without being erroneous also. It is true that the constant (a), if correctly assumed, would have been sufficient; but if this quantity (a) is erroneous, and differs from the quantity (a) which I have adopted, there must be introduced, for the purpose of correcting the original error, a factor (b), such that $ab=a$. This quantity (ab) has unaccountably disappeared in the result of the investigation. If this quantity had been exhibited in the result, inquiries would have been made as to the nature of the quantity (b), which inquiries, probably, could not have been satisfactorily answered.

The formula $y=10^{\frac{k^2a}{\lambda p}(1-p^x)}$, by which the number living or surviving at any age (x) is expressed in terms of the mortality at that age, may be obtained without the aid of the differential calculus, by direct methods, in which no unknown quantities are introduced. The formula may be obtained by a course of investigation similar to that pursued in calculating the amount of £1 with compound interest for one year, when the periods of conversion of interest into capital are moments or equal intervals of time infinitely small. When the rate of interest (or the rate of mortality) is constant, the problem is one of which the solution is familiar to most actuaries. In the case of the rate of interest (or the rate of mortality) continually increasing at a given annual rate (p), the sum in one year

of the momentary increments may be calculated on principles similar to those on which the sum in one year of momentary increments, derived from constant ratios, is commonly calculated. It may be useful here to observe that the series of survivors in tables of mortality resemble, in a great degree, tables of discount representing the values of £1 receivable (x) years hence; that the annual rate of discount of money is $\frac{1}{1+a}$ when the amount of £1, with compound interest for one year, is $(1+a)$; and that, by a change of sign in the exponent, annual ratios of increasing money or population become annual ratios of decreasing money or population, applicable to the same years or intervals of age, in the same table, according to the direction in which progress is measured.

In the first place, let it be supposed that the rate of interest is constant, being such that £1 increases to £$(1+a)$ in every year. Then, let it be required to find another quantity (α), such that $\left(1+\frac{\alpha}{n}\right)^n = 1+a$; the year being divided into an infinite number (n) of equal parts, $\frac{\alpha}{n}$ being a quantity indefinitely small, and $\left(1+\frac{\alpha}{n}\right)$ being a constant ratio repeated or multiplied (n) times in order to produce a result equal to $(1+a)$. In treatises on algebra, it is shown that when $\left(\frac{\alpha}{n}\right)$ is indefinitely small, the quantity $\left(1+\frac{\alpha}{n}\right)=e^{\frac{\alpha}{n}}$, ($e$) being the base of the hyperbolic system of logarithms and equal to 2·7182818. By substituting this value of $\left(1+\frac{\alpha}{n}\right)$ in the first equation, we get $\left(e^{\frac{\alpha}{n}}\right)^n = 1+a$, or $e^{\alpha}=1+a$; that is to say, α = hyperbolic logarithm of $(1+a) = \frac{1}{k} \times$ common logarithm of $(1+a)$; (k) being the modulus of the common system and equal to ·4342945. Similarly, if (p), the annual ratio at which the interest (a) increases, be taken equal to $(1+b)$, and if (β) have the same relation to (b) that (α) has to (a), we shall have

$$\beta = \text{hyp. log.}\,(1+b) = \text{hyp. log.}\,p = \frac{1}{k} \times \text{com. log.}\,p.$$

Let it now be assumed that the rate of interest of money, represented by (a) at the beginning of the year, becomes, by

uniform increase throughout the year, (ap) at the end of the year; and let it be required to ascertain the amount of capital and interest at the end of the year in respect of £1 capital possessed at the beginning of the year. It will be seen that the rate of interest of money, which is $\frac{a}{n}$ at the beginning of the first moment of the year, becomes $\frac{a}{n}\left(1+\frac{\beta}{n}\right)$ at the end of the first of the (n) equal intervals into which the year has been divided, and $\frac{a}{n}\left(1+\frac{\beta}{n}\right)^2$ at the end of the second of these equal intervals; for $p=\left(1+\frac{\beta}{n}\right)^n$, and $\left(1+\frac{\beta}{n}\right)$ is the constant ratio borne by the rate of interest at the end of one moment to the rate of interest which existed at the end of the preceding moment. We shall, consequently, have, for the rates of interest existing at the end of the 1st, 2nd, 3rd, to nth moments—

$$\frac{a}{n}\left(1+\frac{\beta}{n}\right),\ \frac{a}{n}\left(1+\frac{\beta}{n}\right)^2,\ \frac{a}{n}\left(1+\frac{\beta}{n}\right)^3,\ \ldots\ \frac{a}{n}\left(1+\frac{\beta}{n}\right)^n.$$

And, since the ratio of capital and interest together, at the end of any moment, to the capital at the beginning of such moment, is that of (1 + rate of interest) to 1, we shall have for such ratios, in the 1st, 2nd, 3rd, to nth of the (n) infinitely small equal intervals of the year—

$$\left[1+\frac{a}{n}\left(1+\frac{\beta}{n}\right)\right],\ \left[1+\frac{a}{n}\left(1+\frac{\beta}{n}\right)^2\right],\ \left[1+\frac{a}{n}\left(1+\frac{\beta}{n}\right)^3\right]\ \ldots\ \left[1+\frac{a}{n}\left(1+\frac{\beta}{n}\right)^n\right],$$

which may be written thus, on putting $\frac{a}{n}=a'$, and $\left(1+\frac{\beta}{n}\right)=v$:—

$$(1+a'v),\ (1+a'v^2),\ (1+a'v^3)\ \ldots\ldots\ (1+a'v^n);$$

which last quantities may be written as follows, since $a'v$, $a'v^2$, $a'v^3$, &c., are indefinitely small—

$$e^{a'v},\ e^{a'v^2},\ e^{a'v^3}\ \ldots\ldots\ e^{a'v}.$$

The product of all the above factors, (n) in number, or the value of (y) at the end of the year, will be—

$$y=e^{a'v}\times e^{a'v^2}\times e^{a'v^3}\times \text{ \&c. \&c. } \times e^{a'v^n},$$
$$=e^{a'(v+v^2+v^3+\text{ \&c. }+v^n)}=e^{a'\frac{v^n-1}{v-1}\times v}.$$

On taking the hyperbolic logarithms of both sides, we get—

$$\text{hyp. log.}\, y = a' \frac{v^n - 1}{v - 1} \times v = \frac{a}{n} \times \frac{\left(1 + \frac{\beta}{n}\right)^n - 1}{1 + \frac{\beta}{n} - 1} \times \left(1 + \frac{\beta}{n}\right)$$

$$= \frac{a}{\beta}\left(e^{\beta} - 1\right), \text{ omitting } \frac{\beta}{n} \text{ as indefinitely small in } \left(1 + \frac{\beta}{n}\right),$$

$$= \frac{a}{\beta}\left(p - 1\right),$$

which is of the same form as the value previously obtained (p. 180) of the constant used in multiplying (p^x) in order to yield the hyperbolic logarithm of the probability of living one year from the age (x) years.

The value just obtained of the logarithm of the amount of £1 at the end of the first year, when the rate of interest increases by equal proportions in equal infinitely small intervals of time, will, when multiplied successively by $p^1, p^2, p^3, p^4 \ldots . p^{x-1}$, give the values of the logarithms of (y) for the 2nd, 3rd, 4th . . . and xth year. The numbers corresponding to these logarithms will severally represent, for successive years, the capital and interest at the end of each year in respect of £1 capital at the beginning of such year. The multiplication together of all the annual ratios, beginning with the first and ending with the (x)th year, will give the amount of £1 in (x) years when the rate of interest is (a) at the beginning and (ap^x) at the end of the term. If (a) had been taken to represent decrement by mortality, instead of increment by interest of money, a formula in the same terms as that just given would have been obtained, with the difference only of a change of sign from positive to negative in the exponent of $(1 + a'v)$.

The formula now obtained by the most direct method of investigation is in exact agreement with the formula which I have elsewhere obtained by the aid of the differential calculus, neither of them containing any unknown constant like (b), introduced by Mr. Gompertz in the process by which his formula has been obtained. Hence it may be concluded that such a quantity does not belong to the correct formula, and that Mr. De Morgan is not wrong in designating it as superfluous and useless. If Mr. Gompertz has used this unknown quantity in order to obtain a correct result, he must have previously committed an error in the process of integration, of which (b) represents the correction.